Albion College
Distinguished Scholars Program
2018 Mathematics Challenge

The following problems are taken from recent contests in which the Albion math team competes each year. The **M**ichigan **A**utumn **T**ake-**H**ome Challenge is held in November and the Lower Michigan Mathematics Competition in April. Both contests match teams from small colleges; the most recent MATH Challenge included schools from nine states, while the LMMC is for Michigan colleges only. Albion is a three-time LMMC champion, winning that title in 2004, 2007, and 2010; and has also won the MATH Challenge once, in 2001.

The rules for these competitions are the same: Students work on a set of 10 problems in teams of two or three, with a three-hour time limit and no access to books, calculators, notes, the Internet, or other outside assistance. While we don't expect you to abide by the time limit, nor to find two teammates, we do ask that you respect the "no outside assistance, calculators, or computers" rule of the competition. You should be prepared to present your solutions to at least *four* of these problems during the DSP presentation session on February 3, 2018.

Full solutions, with justification, are expected. Simply stating an answer is not sufficient; you must offer a proof or explanation for your results.

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1. Given any 5 points on a sphere, show that there is a closed hemisphere (a hemisphere including its boundary circle) that contains at least four of them.

2. Let $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

   Consider the following operations on matrices:

   (a) Multiply a row or a column by ± 1.

   (b) Add one row to another row, or add one column to another column (of the same matrix—that is, you cannot add one row or column of A to a row or column of B).

   (c) Interchange any two rows or any two columns of a single matrix.

   Does there exist a finite sequence of these operations that transforms matrix A into matrix B?

3. Evaluate
$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}.$$

4. Alice tosses 99 fair coins and Bob tosses 100. What is the probability that Bob gets more (meaning: strictly more) heads than Alice?

5. (a) Find the area of the triangle bounded by the three lines $x = 0$, $y = 0$, and the tangent line to the curve $y = \frac{1}{x}$ at the point $x = 1$.
   (b) Find the area of the triangle bounded by the three lines $x = 0$, $y = 0$, and the tangent line to the curve $y = \frac{1}{x}$ at the point $x = 2$.
   (c) Show that you get the same area using the tangent line to $y = \frac{1}{x}$ at any $x = a > 0$.

6. Recall that for a nonempty finite set $S = \{x_1, x_2, \ldots, x_n\}$ with mean

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

the standard deviation of S is defined to be

$$\sigma = \sqrt{\frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + \cdots + (x_n - \overline{x})^2}{n}}.$$

Prove or disprove: For any nonempty finite set S of positive real numbers, the standard deviation of S cannot be larger than the mean of S.

7. Suppose that f, f', and f'' are continuous functions defined on the interval [0,3] satisfying $f(3) = 4$, $f'(3) = -1$, and $\int_0^3 f(x)dx = 6$. Find the value of

$$\int_0^3 x^2 f''(x)dx.$$