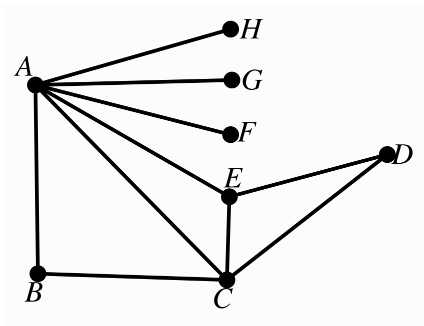
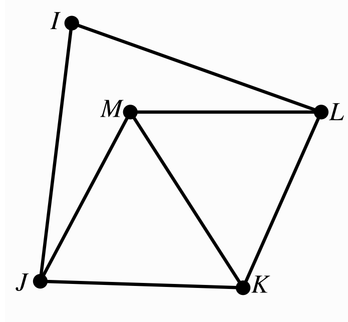
A Graphical Exploration

This team theme will give you the chance to explore some new ideas in mathematics. First let's have some new definitions.

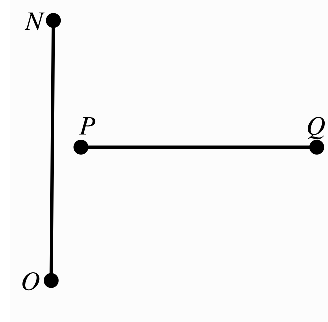
A *graph* is a set of *vertices* and *edges*. We often represent the vertices as points and the edges as segments that connect points. Here are some examples of graphs.



Graph 1



Graph 2



Graph 3

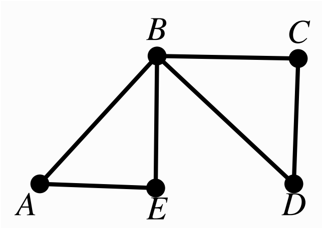
Graph 1 has 8 vertices and 10 edges. Graph 2 has 5 vertices and 7 edges. Graph 3 has 4 vertices and 2 edges.
The *degree* of a vertex is the number of edges connected to a vertex.
Vertex A in graph 1 has degree 6. Vertex B has degree 2.

In graph 2

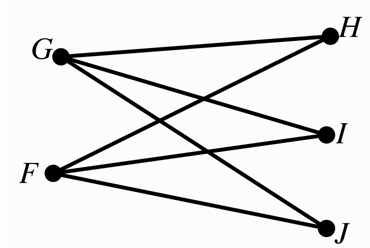
Vertex	I	J	K	L	M
Degree	2	3	3	3	3

the total degree is 14

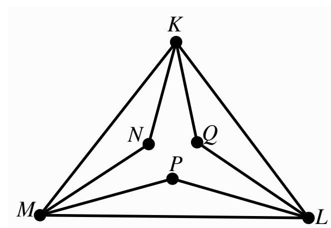
Question 1: Complete tables like the one above for graphs 4, 5, 6.
Then find the total degree of all of the vertices:



Graph 4



Graph 5



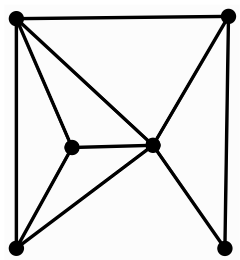
Graph 6

Now count the number of edges in graphs 4, 5, and 6. Notice that the total degree of each graph is twice the number of edges.

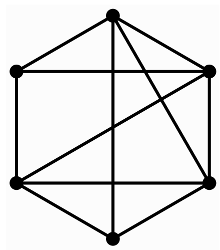
Question 2: Explain why the total degree of a graph is twice the number of edges.

When you found the degree of each vertex in a graph you may have noticed that some of the degrees were odd and some were even.

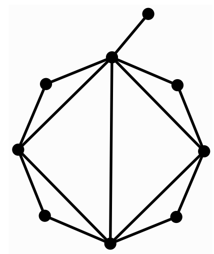
Question 3: Find the degree of each vertex in each of the graphs below. How many vertices have an odd degree?



Graph 7



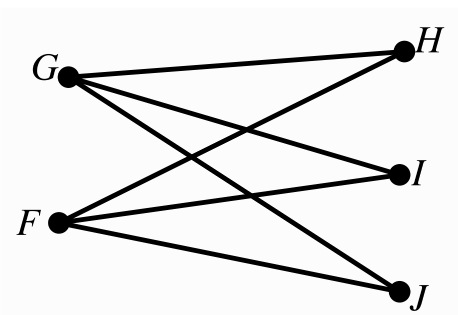
Graph 8



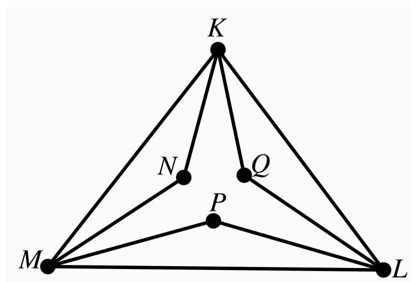
Graph 9

Question 4: Is it possible for a graph to have an odd number of vertices with odd degree? Explain your answer.

A *path* from one vertex to another vertex in a graph is a sequence of adjacent vertices and edges of the graph. In a path you can pass through each vertex only once and use each edge only once. Depending on the graph, there may be more than one path between two different vertices. In graph 5 there are three different paths from vertex G to vertex H. The shortest path is to go directly from G to H. Another path is to go from G to I to F to H. The final possibility is to go from G to J to F to H.



Graph 5



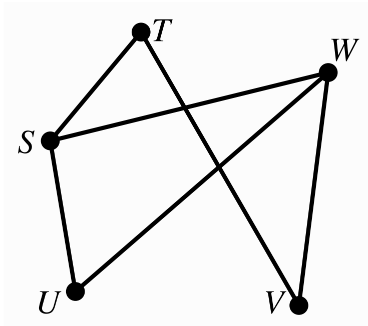
Graph 6

Question 5: How many paths are there from vertex H to vertex J in graph 5? List the paths in order from shortest to longest. Which pairs of vertices in graph 5 have paths between them that pass through every vertex? Which pairs do not have such a path between them?

Question 6: Which pairs of vertices in graph 6 have paths between them that pass through every vertex? Which do not? Explain what features of the graph makes the difference.

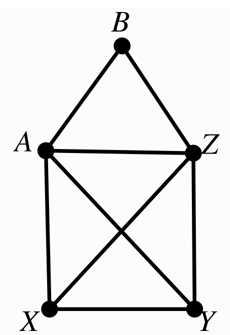
Question 7: In graph 6 there were 6 vertices. There were some pairs of vertices where there were no paths between them that passed through every other vertex. Draw your own graph with 6 vertices and include the minimum amount of edges such that there is a path from any vertex to any other vertex passing through every vertex. Explain how you know your graph has the fewest edges necessary to make this happen.

A *trail* is like a path, but you are allowed to pass through a vertex more than once. An *Euler trail* is a trail that passes over all of the edges exactly once. It is named for Leonhard Euler, the great Swiss mathematician who solved some early problems with graphs. In graph 10 there is an Euler trail from vertex S to vertex W. Start at vertex S, then go to U, then to W, next go to V, then up to T, back to S, and finally end at W. You have covered each edge exactly once.

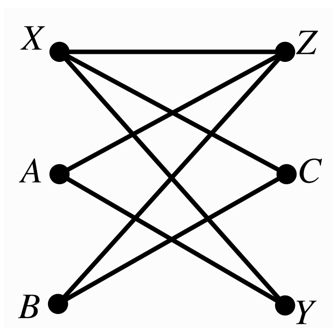


Graph 10

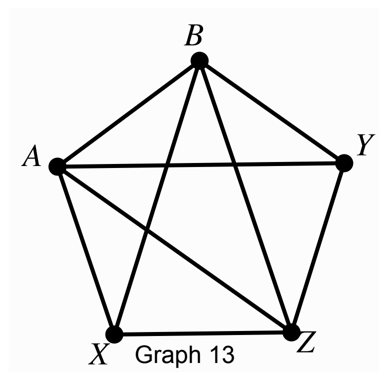
Question 8: In each of the graphs below, try to find an Euler trail from X to Y. Then try to find an Euler trail from X to Z. If you can find a trail, clearly describe it. If you cannot find an Euler trail, explain why it is not possible.



Graph 11



Graph 12

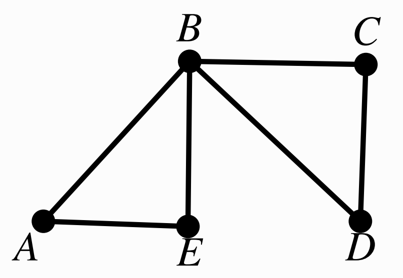


Graph 13

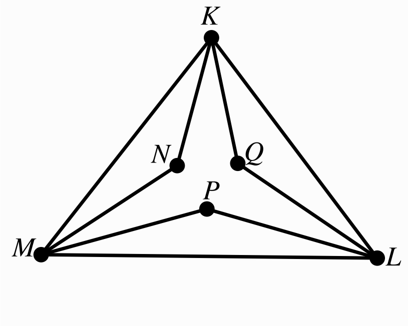
Question 9: Are there Euler trails between any other pairs of vertices in any of graphs 11, 12, or 13? If so, state where the trails start and where they end.

Question 10: What determines whether or not a graph has an Euler trail? Explain your reasoning.

A trail that begins and ends at the same vertex is called a *circuit*. A circuit that uses every edge exactly once is called an *Euler circuit*. You may reuse vertices if you wish, but you cannot reuse edges. In graph 4 a circuit that begins and ends at vertex B is completed by passing through the vertices in this order: $BCDBAEB$. Finding an Euler circuit is the same thing as drawing the graph without retracing any of the edges, starting and ending at the same point. Is there an Euler circuit for graph 6?

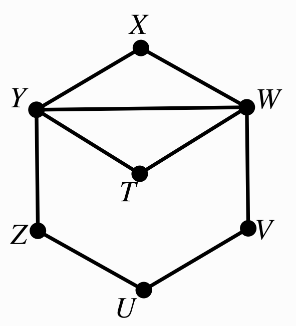


Graph 4

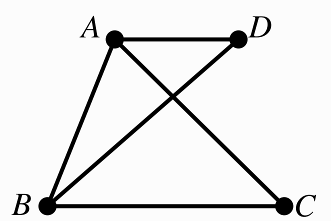


Graph 6

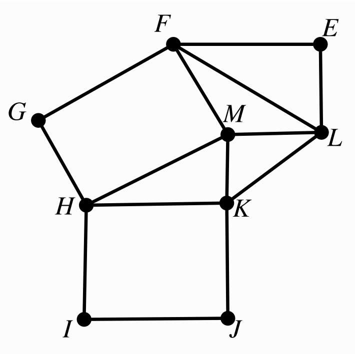
Question 11: If a graph below has an Euler circuit, describe it by listing the vertices in order for the circuit. If it does not have an Euler circuit, explain why. Think about how the degrees of the vertices have an effect on whether or not you can find an Euler circuit. Remember, a circuit has to start and end at the same vertex.



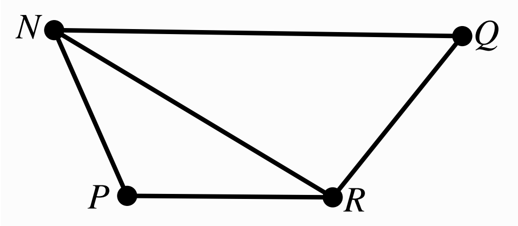
Graph 14



Graph 15



Graph 16



Graph 17

Answers to Team Theme Questions

Question 1:

Graph 4:

Vertex	A	B	C	D	E
Degree	2	4	2	2	2

total degree = 12

Graph 5:

Vertex	F	G	H	I	J
Degree	3	3	2	2	2

total degree = 12

Graph 6:

Vertex	K	L	M	N	P	Q
Degree	4	4	4	2	2	2

total degree = 18

Question 2: Each edge is incident on two vertices. When determining the degree of a vertex, you add up the number of edges incident on it. So each edge is counted twice, once at each of the two vertices it is incident on. Thus the total degree is twice the number of edges.

Question 3:

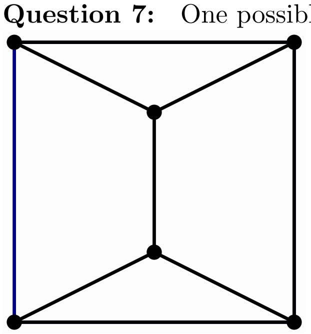
Graph 7: Degrees of vertices are $2, 3, 3, 3, 4, 5$. Four vertices have odd degree in graph 7.
Graph 8: Degrees of vertices are $3, 3, 4, 4, 4, 4$. Two vertices have odd degree in graph 8.
Graph 9: Degrees of vertices are $1, 2, 2, 2, 2, 4, 4, 5$. Two vertices have odd degree in graph 9.

Question 4: It is not possible of a graph to have an odd number of vertices with odd degree. The total degree of a graph is even since it is twice the number of edges. Suppose we let the sum of the degrees of the even vertices be represented by E and the sum of the degrees of the odd vertices be represented by D. Then $E + D$ is the total degree. We know this is an even number. Because E is the sum of even numbers is must be even. So we are adding an even number plus D and getting an even number. This means that D must also be even. Each odd vertex has odd degree. In order to get an even total, we must add up an even number of odd numbers. So there but be an even number of odd vertices.

Question 5: There are four paths from vertex H to vertex J. In order from shortest to longest these paths are:
HGJ, HFJ
$HGIFJ, HFIGJ$
There are paths between any pair of vertices on the right side of the graph that pass through every vertex. If you choose a pair of vertices on the left side or a pair that contains one vertex on the left and one vertex on the right, there is no path between them that passes through every vertex.

Question 6: The pairs that have paths between them passing through every vertex are the following:
K and L, M, Q, N,
L and K, M, Q, P,
M and K, L, N, P,
N and K, M, P, Q,
P and L, M, N, Q,
Q and K, L, N, P.
The pairs that do not have such paths are K and P, L and N, and M and Q.
When trying to find a path from M to Q there are four options starting from M. Consider going from M to N. From here there is only one option, going on to K. Then in order to not go to Q immediately, you must go to L. At this point if you go to P, there is no way to get back to Q without repeating a vertex. The same thing happens if you start from M and go to P. Note that starting from M and going straight to either K or L immediately eliminates the possibility of going through N or P so these starting choices are not really an option. The difference with going from M to P or N is that there is a path around the inside of the graph that passes through every vertex and these pairs of vertices are adjacent in this path, where M and Q are not.

Question 7: One possible graph is shown below. It has 9 edges.



Using an arrangement like this, there is always an option, after arriving at a vertex, of two different edges to follow. That allows you to complete a path from any vertex to any other vertex and pass through all vertices on the way. This is the minimum number of edges that makes this possible.

Question 8: In graph 11 a possible Euler trail from X to Y is $XYABZXAZY$. An Euler trail from X to Z is not possible. Every time you try to do this, you wind up with one edge that is not covered. This is because the degree of Z is even. Along the trail you will take one edge in to Z, one edge out, a third edge back in, but you will need to leave Z a second time in order to use all of the edges incident on Z. In graph 12 an Euler trail from X to Y is not possible. Because the degree of vertex Y is even, you can't possible end your trail there. You must follow one edge to Y and then use the other edge to leave Y. An Euler trail from X to Z is $XCBXYAZ$. In graph 13 an Euler trail from X to Y is $XAZXBAYZBY$. An Euler trail from X to Z is not possible. This is the same case as in the other graphs where a trail is impossible. The proposed final vertex has an even degree so to use all of the edges incident on this vertex you would need to enter and leave the vertex twice, thus not being able to end the trail there.

Question 9: There are no other pairs of vertices that have Euler trails between them. This is because there are no other pairs of vertices that have odd degrees.

Question 10: If a graph has exactly two vertices with odd degree, then there is an Euler trail between those vertices. If there are more than two vertices with odd degree then it will be impossible to create an Euler trail. To traverse all of the edges that are incident to an odd degree vertex, you need to leave, enter, and leave the vertex again, or enter, leave, and reenter. (If the degree is higher than three, you will need to repeat the enter and leave cycle more times.) The first of these takes place at the initial vertex and the other at the final vertex. At any other vertex of odd degree you will need to omit one edge in order to complete a trail. If there are fewer than two vertices with odd degree then you will not be able to complete an Euler circuit. We know that there must be an even number of vertices with odd degree. So if there are fewer than two, then there must be none. If you start a trail at a vertex of even degree, then you will eventually become stuck at that vertex when you use the last edge incident to it. Thus there must be exactly two vertices of odd degree in order for the graph to have an Euler trail.

Question 11: Graph 14 has an Euler circuit $XYWTYZUVWX$. Graph 15 does not have an Euler circuit. Vertices A and B both have odd degree so while there is an Euler trail between them, if you start at A, there will be no way to return to A after leaving it a second time as is necessary in order to use all of the edges. Graph 16 has an Euler circuit $EFGHIJKHMKLMFLE$. Graph 17 does not have an Euler circuit. It has two vertices with odd degree, vertex N and vertex R. So there is an Euler trail between these two vertices, but you cannot complete a circuit. Starting at either of these vertices, you must use all of the edges incident on that vertex. This means that you leave the vertex, return, and leave again. There is no way to come back to complete the circuit without reusing an edge. If you try and start a circuit at an even degree vertex, you must wind up stuck at an odd degree vertex as you go to that vertex, leave, and return in order to use all of the edges. There is no remaining edge that allows you to leave the odd vertex a final time. Thus to have an Euler circuit, all of the vertices must have even degree.